Solving for S(t) and E[S(t)] in Geometric Brownian Motion

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1 Solving for S(t)

Geometric Brownian Motion satisfies the familiar SDE:

$$dS(t) = S(t)[\mu dt + \sigma dW(t)] \tag{1}$$

$$S(0) = s \tag{2}$$

In order to solve for *S(t)* we will apply Ito to dln*S(t)*:

$$dlnS(t) = \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S(t)^2}dS(t)^2 \tag{3}$$

$$= \frac{1}{S(t)}S(t)[\mu dt + \sigma dW(t)] - \frac{1}{2}\frac{1}{S(t)^2}S(t)^2[\sigma^2 dW(t)^2] \tag{4}$$

$$dlnS(t) = \mu dt + \sigma dW(t) - \frac{1}{2}\sigma^2 dt \tag{5}$$

Then we integrate and apply the fundamental theorem of calculus to get:

$$lnS(t) - lnS(0) = (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \tag{6}$$

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)} \tag{7}$$

2 Solving for E[S(t)]

We now take the expectation of the expression in equation (7):

$$\mathbb{E}[S(t)] = \mathbb{E}[S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}] \tag{8}$$

Recall the general formula for the expected value of a Gaussian random variable:

$$\mathbb{E}[e^X] = \mathbb{E}[e^{\mu + \frac{1}{2}\sigma^2}] \tag{9}$$

where X has the law of a normal random variable with mean μ and variance σ^2. We know that Brownian Motion $\sim N(0,$ t). Applying the rule to what we have in equation (8) and the fact that the stock price at time 0 (today) is known we get:

$$\mathbb{E}[S(t)] = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t}\mathbb{E}[e^{\sigma W(t)}] \tag{10}$$

$$= S(0)e^{(\mu - \frac{1}{2}\sigma^2)t}e^{0 + \frac{1}{2}\sigma^2 t} \tag{11}$$

$$\mathbb{E}[S(t)] = S(0)e^{\mu t} \tag{12}$$