Solving the Heat Equation With Green's Function

Ophir Gottlieb

3/21/2007

1 Setting Up the Problem

The general heat equation with a heat source is written as:

$$u_t(x,t) = k\Delta u(x,t) + g(x,t) \quad in\ \Omega \tag{1}$$

$$u(x,0) = f(x,t) \quad in\ \Omega \tag{2}$$

$$u(x,t) = h(x,t) \quad on\ \partial\Omega \tag{3}$$

Where $x \leq x_0 \in \Omega$, $0 < t \leq t_0$

Further, we know that G(x,t) satisfies:

$$G_t = -k\Delta G \tag{4}$$

$$G(x,t) = 0\ on\ \partial\Omega \tag{5}$$

$$G(x,t_0) = \delta(x - x_0) \tag{6}$$

2 Finding the Solution with Green's Function

We begin by taking equation (1), multiplying by G(x,t) on both sides and integrating on both sides over the volume integral and over time to get:

$$\int_0^{t_0}\int\int_\Omega (G\cdot u_t)dxdt = \int_0^{t_0}\int\int_\Omega (G\cdot k\Delta u)dxdt + \int_0^{t_0}\int\int_\Omega (G\cdot g)dxdt \tag{7}$$

We can re-write the LHS integral as:

$$\int_0^{t_0}\int\int_\Omega (G\cdot u_t)dxdt = \int\int_\Omega (G\cdot u)|_{t=0}^{t=t_0}dx - \int_0^{t_0}\int\int_\Omega (G_t\cdot u)dxdt \tag{8}$$

We can re-write the RHS after integrating buy parts twice to get:

$$RHS = \int_0^{t_0} \int \int_\Omega (ku)\Delta G dxdt + \int_0^{t_0} \int_{\partial\Omega} kG\frac{\partial u}{\partial n} dS(x)dt - \int_0^{t_0} \int_{\partial\Omega} ku\frac{\partial G}{\partial n} dS(x)dt + \int_0^{t_0} \int \int_\Omega (G \cdot g) dxdt \tag{9}$$

We notice that we have both terms of equation (4) so we can cancel term 2 on the LHS with term 1 on the RHS. We further know that G=0 on $\partial\Omega$ which allows us to cancel the second term on the right hand side. Specifically we have:

$$\int_0^{t_0} \int \int_\Omega (ku)\Delta G dxdt + \int_0^{t_0} \int \int_\Omega (G_t \cdot u) dxdt = 0 \quad \textit{by equation } (4) \tag{10}$$

$$\int_0^{t_0} \int_{\partial\Omega} kG\frac{\partial u}{\partial n} dS(x)dt = 0 \quad \textit{by equation } (5) \tag{11}$$

Finally we are left with:

$$\int \int_\Omega (G \cdot u)|_{t=0}^{t=t_0} dx = -\int_0^{t_0} \int_{\partial\Omega} ku\frac{\partial G}{\partial n} dS(x)dt + \int_0^{t_0} \int \int_\Omega (G \cdot g) dxdt \tag{12}$$

We now write the LHS as:

$$\int \int_\Omega (G \cdot u)|_{t=0}^{t=t_0} dx = \int \int_\Omega u \cdot \delta(x - x_0) dx - \int \int_\Omega f \cdot G(x, x_0; 0, t_0) dx \tag{13}$$

Which by the property of the delta function yields:

$$= u(x_0, t_0) - \int \int_\Omega f \cdot G(x, x_0; 0, t_0) dx \tag{14}$$

Finally, putting it all together (and substituting h(x,t) for u(x,t) when we integrate over $\partial\Omega$) we find the solution formula to the general heat equation using Green's function:

$$u(x_0, t_0) = \int \int_\Omega f \cdot G(x, x_0; 0, t_0) dx - \int_0^{t_0} \int_{\partial\Omega} k \cdot h\frac{\partial G}{\partial n} dS(x)dt + \int_0^{t_0} \int \int_\Omega G \cdot g dxdt \tag{15}$$

This motivates the importance of finding Green's function for a particular problem, as with it, we have a solution to the PDE. You can see the "Heat Equation: Solving for Green's Function" article in order to complete the analysis.