Deriving the Heat Kernel in 1 Dimension

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1 Set Up - Shifting the Data

The general heat equation without a heat source is written as:

$$u_t(x,t) = k\Delta u(x,t) \tag{1}$$

Further, denoting the heat kernel G(x,t) we take the conditions:

$$G_t = k\Delta G \tag{2}$$

$$G(x,0) = \delta(x) \tag{3}$$

Where δ(x) has the following properties:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{4}$$

$$\delta(x) = 0,\ x \neq 0 \tag{5}$$

$$\delta(\alpha x) = \frac{1}{\alpha}\delta(x) \tag{6}$$

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0) \tag{7}$$

Our goal is to find the Kernel G(x,t). We now write:

$$H(x,t) = \alpha G(\beta x, \gamma t) \tag{8}$$

With α, β, and γ to be determined. Now let's try to select the appropriate relationship between α, β γ so that H and G satisfy the same equations. We have from equation (3) and (6) that:

$$H(x,0) = \alpha G(\beta x, 0) = \alpha\delta(\beta x) = \frac{\alpha}{\beta}\delta(x) \tag{9}$$

Therefore if $\alpha = \beta$ then H(x,0) = G(x,0) = δ(x). We now want H(x,t) to satisfy equation (2).

$$H_t = \alpha\gamma G_t \tag{10}$$

$$H_{xx} = \alpha\beta^2 G_{xx} \tag{11}$$

Since we want H(x,t) to also satisfy equation (2) we have:

$$\alpha\gamma G_t = \alpha\beta^2 G_{xx} \tag{12}$$

And therefore $\gamma = \beta^2$. This together gives us that if $\alpha = \beta$ and $\gamma = \beta^2$, then H(x,t) and G(x,t) both satisfy the conditions for the heat kernel, and therefore by uniqueness of the heat kernel H(x,t) $\equiv$ G(x,t). We have successfully found the requirements to correctly shift the data and can write:

$$G(x,t) = \alpha(G\beta x, \gamma t) \tag{13}$$

2 Solving for the Heat Kernel

Now we choose a clever value for γ and get:

$$\gamma = \frac{1}{t} \tag{14}$$

$$\alpha = \beta = \frac{1}{\sqrt{t}} \tag{15}$$

Now we write:

$$G(x,t) = \frac{1}{\sqrt{t}}G(\frac{x}{\sqrt{t}}, 1) = \frac{1}{\sqrt{t}}Q(\epsilon) \tag{16}$$

$$\epsilon = \frac{x}{\sqrt{t}} \tag{17}$$

We can now derive an ODE for Q(ϵ) using equation (2):

$$G_t = -\frac{1}{2}t^{-\frac{3}{2}}Q \; - \; \frac{1}{2}t^{-\frac{3}{2}}(t^{-\frac{1}{2}}x)Q' \tag{18}$$

$$= -\frac{1}{2}t^{-\frac{3}{2}}Q - \frac{1}{2}t^{-\frac{3}{2}}\epsilon Q' \tag{19}$$

$$G_t = -\frac{1}{2}t^{-\frac{3}{2}}(Q + \epsilon Q') \tag{20}$$

$$G_{xx} = t^{-\frac{3}{2}}Q'' \tag{21}$$

By equation (2) we get:

$$-\frac{1}{2}t^{-\frac{3}{2}}(Q + \epsilon Q') = kt^{-\frac{3}{2}}Q'' \tag{22}$$

$$Q + \epsilon Q' + 2kQ'' = 0 \tag{23}$$

$$(\epsilon Q)' + 2kQ'' = 0 \tag{24}$$

Integrating we get:

$$\epsilon Q + 2kQ' = c \tag{25}$$

Now we note that we expect G(∞, t) = 0 $\forall$t. Consequently we expect Q$'$ = 0 as x $\rightarrow$ ∞. Therefore c = 0. We then have the ODE:

$$\epsilon Q + 2kQ' = 0 \tag{26}$$

$$2kQ' = -\epsilon Q \tag{27}$$

$$Q' = -\frac{\epsilon Q}{2k} \tag{28}$$

$$\frac{Q'}{Q} = -\frac{\epsilon}{2k} \tag{29}$$

Integrating and simplifying we get:

$$\ln Q = -\frac{\epsilon^2}{4k} + C \tag{30}$$

$$Q(\epsilon) = Ce^{-\frac{\epsilon^2}{4k}} \tag{31}$$

Substituting equation (16) for Q(ϵ) and equation (17) for ϵ we get:

$$G(x,t) = \frac{1}{\sqrt{t}} C e^{-\frac{x^2}{4kt}} \tag{32}$$

We know G(x,0) = δ(x) therefore:

$$\int_{-\infty}^{\infty} G(x,0)dx = 1 \tag{33}$$

$$and \ \frac{d}{dt}\int_{-\infty}^{\infty} G(x,t)dx = k\int_{-\infty}^{\infty} G_{xx}dx = 0 \ \ by \ \ the \ \ conservation \ \ of \ \ energy \tag{34}$$

$$therefore \ \ \int_{-\infty}^{\infty} G(x,t)dx = 1 \tag{35}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} C e^{-\frac{x^2}{4kt}} dx \tag{36}$$

$$= \sqrt{4k} C \int_{-\infty}^{\infty} e^{-x^2} dx \tag{37}$$

$$= \sqrt{4k} C \sqrt{\pi} = 1 \tag{38}$$

$$\Rightarrow C = \frac{1}{\sqrt{4k\pi}} \tag{39}$$

And finally:

$$G(x,t) = \frac{1}{\sqrt{4kt\pi}} e^{\frac{-x^2}{4kt}} \tag{40}$$