Deriving the Binomial Tree Risk Neutral Probability and Delta

Ophir Gottlieb

10/11/2007

1 Set Up

Using risk neutral pricing theory and a simple one step binomial tree, we can derive the risk neutral measure for pricing. From this measure, it is an easy extension to derive the expression for delta (for a call option).

We let S_t be the stock price at time t. In this simple example we have only two time periods; $t = 0$ and $t = 1$. We can write the starting stock price as S_0 with only the two possibilities of going up to $u \cdot S_0$ or down to $d \cdot S_0$; where $u > 1.0$ and $d < 1.0$. Whether we move up or down from S_0 we write the resulting price as S_1.

Next we define the interest rate for the single period to be r and assume continuous discounting. Finally we define the *risk-neutral* probabilities of moving up or down as q_u and q_d. The simple set up is illustrated below.

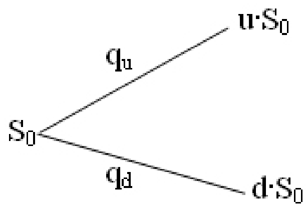

Figure 1: Binomial Tree Setup For Underlying Stock

2 Find the Risk Neutral Measure

Our first goal is to find a closed form solution for the risk neutral probabilities. Starting from the theory of risk-neutral pricing and denoting q as the risk neutral measure we can write:

$$S_0 = e^{-r\cdot 1}\mathbb{E}^q[S_1] \tag{1}$$

Expanding S_1:

$$S_0 = e^{-r}\mathbb{E}^q[\cdot S_0 \cdot u + \cdot S_0 \cdot d]$$

Evaluating the expectation and simplifying:

$$S_0 = e^{-r}[q_u \cdot S_0 \cdot u + q_d \cdot S_0 \cdot d]$$

$$S_0 \cdot e^r = S_0[q_u \cdot u + q_d \cdot d]$$

$$e^r = [q_u \cdot u + q_d \cdot d]$$

$$e^r = [q_u \cdot u + (1 - q_u) \cdot d]$$

$$e^r = q_u \cdot [u - d] + d$$

Finally yielding our solution:

$$q_u = \frac{e^r - d}{u - d} \tag{2}$$

3 Find the Expression for Delta

Just as we can write the one step binomial tree for the underlying security, we can write it for a call option. With a slight change of notation for convenience we can write the call option price today as C_0 and the call option price after one period as either C_u (if the underlying stock goes up) or C_d (if the underlying stock goes down).

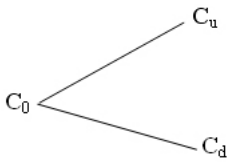

Figure 2: Binomial Tree Setup For Call Option

We know we can replicate a call option with some number of shares (Δ) of stock and borrowing from the bank at interest rate r. Therefore, by the fundamental theorem of finance we know that the price of the call option at each state must be the same as the price of the portfolio at each state. Mathematically we can write this as:

$$\Delta \cdot u \cdot S_0 - C_u = \Delta \cdot d \cdot S_0 - C_d \tag{3}$$

$$\Delta \cdot S_0[u - d] = C_u - C_d$$

Yielding our final expression:

$$\Delta = \frac{C_u - C_d}{S_0[u - d]} \tag{4}$$