Deriving the Black-Scholes PDE Using Risk Neutral Pricing

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1 Set Up

Using risk neutral pricing theory (also called arbitrage free pricing theory) we have yet another way to derive the well known Black-Scholes PDE for an European Option. The fundamental discovery in risk neutral pricing theory states that *every* risky traded asset, when discounted, must be a martingale with respect to the relevant filtration. As always, we assume Geometric Brownian Motion under the risk neutral (Martingale) measure:

$$ds(t) = S(t)[rdt + \sigma dW(t)] \tag{1}$$

$$S(0) = s \tag{2}$$

And we have the discounted option process:

$$e^{-rt}C(t, S(t)) \tag{3}$$

Where r is the risk free interest rate, σ is the volatility, W(t) is the standard Brownian Motion and C(t,S(t)) is the option price suppressing the other arguments for notational convenience. One of the crucial assumptions to general Black-Scholes theory is the assumption that r and σ are deterministic (and in this case, constants).

2 Proving that the discounted option is a Martingale implies the BS PDE

Claim:

$$\mathbb{E}^{Q}[C(t_2, S_{t_2})|\mathcal{F}_{t_1}] = e^{-rt_1}C(t_1, S_{t_1}) \tag{4}$$

Implies:

$$-rC(t,S_t)+C_t+C_sS(t)r+\frac{1}{2}C_{ss}S(t)^2\sigma^2=0 \tag{5}$$

Where the Q indicates we are taking the expectation using the risk neutral measure and $\mathcal{F}_t$ is the relevant increasing collection of sigma algebras. In order to evaluate this expectation, we first apply Ito to the discounted option:

$$d[e^{-rt}C(t,S_t)] \tag{6}$$

$$=d(e^{-rt})C(t,S_t)+e^{-rt}dC \tag{7}$$

$$=-re^{-rt}C(t,S_t)+e^{-rt}[C_tdt+C_sdS+\frac{1}{2}C_{ss}(dS)^2] \tag{8}$$

$$=-re^{-rt}C(t,S_t)+e^{-rt}[C_tdt+C_sS(t)(rdt+\sigma dW(t))+\frac{1}{2}C_{ss}S(t)^2\sigma^2] \tag{9}$$

$$=e^{-rt}[-rC(t,S_t)+C_t+C_sS(t)r+\frac{1}{2}C_{ss}S(t)^2\sigma^2]dt+e^{-rt}C_sS(t)\sigma dW(t) \tag{10}$$

$$d[e^{-rt}C(t,S_t)]=e^{-rt}[BSPDE]dt+e^{-rt}C_sS(t)\sigma dW(t) \tag{11}$$

Where BSPDE is the Black Scholes PDE. Now, by the fundamental theorem of calculus we get:

$$e^{-rt_2}C(t_2,S_{t_2})-e^{-rt_1}C(t_1,S_{t_1})=\int_{t_1}^{t_2}BSPDEdt+\int_{t_1}^{t_2}e^{-rt}C_sS(t)\sigma dW(t) \tag{12}$$

$$e^{-rt_2}C(t_2,S_{t_2})=e^{-rt_1}C(t_1,S_{t_1})+\int_{t_1}^{t_2}BSPDEdt+\int_{t_1}^{t_2}e^{-rt}C_sS(t)\sigma dW(t) \tag{13}$$

And now we take expectation with respect to the risk neutral measure:

$$\mathbb{E}^Q[e^{-rt_2}C(t_2, S_{t_2})|\mathcal{F}t_1] = \mathbb{E}^Q[e^{-rt_1}C(t_1, S_{t_1})|\mathcal{F}t_1] + \mathbb{E}^Q[\int_{t_1}^{t_2} BSPDEdt|\mathcal{F}t_1] + \mathbb{E}^Q[\int_{t_1}^{t_2} e^{-rt}C_sS(t)\sigma dW(t)|\mathcal{F}t_1] \tag{14}$$

On the right hand side, the first expectation is measurable on the filtration and the third expectation is over an Ito integral which itself is a zero mean Martingale. This yields:

$$\mathbb{E}^Q[e^{-rt_2}C(t_2, S_{t_2})|\mathcal{F}t_1] = e^{-rt_1}C(t_1, S_{t_1}) + \mathbb{E}^Q[\int_{t_1}^{t_2} BSPDEdt|\mathcal{F}t_1] + 0 \tag{15}$$

And finally we see that we can only satisfy the Martingale conditional expectation property if the term inside the integral is in fact zero. Therefore we have:

$$e^{-rt}[-rC(t, S_t) + C_t + C_sS(t)r + \frac{1}{2}C_{ss}S(t)^2\sigma^2]dt = 0 \tag{16}$$

Canceling the e^{-rt} and dt factors (neither of which can be zero) we arrive at the Black Scholes PDE:

$$-rC + C_t + C_sSr + \frac{1}{2}C_{ss}S^2\sigma^2 = 0 \tag{17}$$