Deriving Put-Call Parity

Ophir Gottlieb

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1 Put Call Parity

The put-call parity relationship comes nicely from some simple but clever steps. The analysis begins with following true expression:

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K \tag{1}$$

Where $T < t$ is the expiration time of the options. Re-arranging we get:

$$(S_T - K)^+ + K = (K - S_T)^+ + S_T \tag{2}$$

Now we multiply each side by the discount factor $e^{-r(T-t)}$ with t< T:

$$e^{-r(T-t)}(S_T - K)^+ + e^{-r(T-t)}K = e^{-r(T-t)}(K - S_T)^+ + e^{-r(T-t)}S_T \tag{3}$$

Take conditional expectations under the risk neutral measure with respect to the stock price at some time t < T:

$$\begin{aligned}\mathbb{E}^Q[e^{-r(T-t)}(S_T - K)^+|S_t = s] + \mathbb{E}^Q[e^{-r(T-t)}K|S_t = s] = \\ \mathbb{E}^Q[e^{-r(T-t)}(K - S_T)^+|S_t = s] + \mathbb{E}^Q[e^{-r(T-t)}S_T|S_t = s]\end{aligned} \tag{4}$$

Now we recall that risk neutral pricing theory tells us that the discounted value of a risky asset (any risky and traded asset) is a Martingale. This immediately gives us that the first expectation is the price of a Call option at time t, the first expectation on the RHS of the equality is the price of a Put option at time t, and the second expectation on the right hand side is the price of the stock at time t. Finally the second expectation on the LHS is simply a deterministic function and therefore the expectation goes away. This yields the put-call parity relationship:

$$C_t + e^{-r(T-t)}K = P_t + S_t \tag{5}$$