Introductory Problems in SDE

Ophir Gottlieb

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1. Compute $\mathbb{E}[{W_t}^4]$ by applying Ito's formula to ${W_t}^4$.

$$dW_t^4 = 4W_t^3 dW_t + \frac{1}{2} \cdot 4 \cdot 3 \cdot W_t^2 (dW_t)^2; \;\; (W_t^4(0) = 0) \tag{1}$$

$$= 4W_t^3 dW_t + 6W_t^2 dt$$

Integrate:

$$W_t^4 = 4\int_0^t W_s^3 dW_s + 6\int_0^t W_s^2 ds + 0$$

Take Expectations and apply Fubini to move the expectation inside the integral:

$$\mathbb{E}[W_t^4] = 0 + 6\int_0^t \mathbb{E}[W_s^2] ds$$

$$= 6\int_0^t s ds$$

$$\mathbb{E}[W_t^4] = 3t^2 \tag{2}$$

2. Apply Ito's formula to ${W_t}^2$ to compute:

$$\int_0^t W_S dW_s \tag{3}$$

$$dW_t^2 = 2W_t dW_t + \frac{1}{2} \cdot 2 \cdot dt; \;\; (W_t^2 = 0)$$

Integrate:

$$W_t^2 = 2\int_0^t W_s dW_s + \int_0^t ds$$

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t \tag{4}$$

3. Apply Ito's formula to compute $\mathbb{E}[\mathrm{S}_t/\mathrm{S}_0]$ from Geometric Brownian Motion defined as:

$$dS(t) = S_t[\mu dt + \sigma dW_t] \tag{5}$$

$$S_0 = s \tag{6}$$

In order to solve for $\mathbb{E}[\mathrm{S}_t/\mathrm{S}_0]$ we will apply Ito to $\mathrm{dln}S_t$:

$$dlnS_t = \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}dS_t^2$$

$$= \frac{1}{S_t}S_t[\mu dt + \sigma dW_t] - \frac{1}{2}\frac{1}{S_t^2}S_t^2[\sigma^2 dW_t^2]$$

$$dlnS_t = \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$$

Then we integrate and apply the fundamental theorem of calculus to get:

$$lnS_t - lnS_0 = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

Take the expectations:

$$\mathbb{E}[S_t] = \mathbb{E}[S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}]$$

Recall the general formula for the expected value of a Gaussian random variable:

$$\mathbb{E}[e^X] = \mathbb{E}[e^{\mu + \frac{1}{2}\sigma^2}]$$

where X has the law of a normal random variable with mean μ and variance σ^2. We know that Brownian Motion $\sim N(0, \mathrm{t})$. Applying the rule we get:

$$\mathbb{E}[S_t] = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t} \cdot \mathbb{E}[e^{\sigma W_t}]$$

$$\mathbb{E}[S_t/S_0] = e^{(\mu - \frac{1}{2}\sigma^2)t} \cdot \mathbb{E}[e^{\sigma W_t}]$$

$$\mathbb{E}[S_t/S_0] = e^{\mu t} \tag{7}$$