Find an Explicit Solution for Delta in Black-Scholes

Ophir Gottlieb

11/7/2007

1 Introduction

We have seen through the creation of a replicating portfolio that the delta required to hedge an European call option is simply $\frac{\partial C}{\partial S}$. Now we will explicitly compute delta by differentiating the closed form Black-Scholes Formula once with respect to the underlying stock.

We recall the Black-Scholes formula for an European call option today (*t=0*) expiring at time $t = T$ with constant interest rate (r), constant volatility (σ) and strike price K as:

$$C = S \cdot \Phi(d_1) - e^{-rT} \cdot K \cdot \Phi(d_2) \tag{1}$$

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where $\Phi(.)$ is the standard normal cumulative distribution. We will also write $\phi(.)$ as the standard normal probability density.

2 Finding $\frac{\partial C}{\partial S}$

First we note that by using the chain rule we find:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S} \cdot \frac{1}{\sigma\sqrt{T}}$$

We can then differentiate equation (1) with respect to S to find:

$$\frac{\partial C}{\partial S} = \Phi(d_1) + \frac{1}{S} \cdot \frac{1}{\sigma\sqrt{T}}[S \cdot \phi(d_1) - e^{-rT} \cdot K \cdot \phi(d_2)] \tag{2}$$

We now make the claim that $[S \cdot \phi(d_1) - e^{-rT} \cdot K \cdot \phi(d_2)] = 0$ and are thus are left with the result that $\Delta = \frac{\partial C}{\partial S} = \Phi(d_1)$.

Proof:

Starting with a simple substitution for d_2 and then moving through the algebra:

$$e^{-rT} \cdot K \cdot \phi(d_2) = e^{-rT} \cdot K \cdot \phi(d_1 - \sigma\sqrt{T})$$

$$= e^{-rT} \cdot K \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-(d_1 - \sigma\sqrt{T})^2}{2}}$$

$$= e^{-rT} \cdot K \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-d_1^2}{2}} \cdot e^{\frac{-(-2d_1\sigma\sqrt{T} + \sigma^2 T)}{2}}$$

$$= e^{-rT} \cdot K \cdot \phi(d1) \cdot e^{d_1\sigma\sqrt{T}} \cdot e^{-\sigma^2 T}$$

$$= K \cdot \phi(d1) \cdot e^{-rT - \frac{\sigma^2 T}{2} + ln(\frac{S}{K}) + rT + \frac{\sigma^2 T}{2}}$$

$$= K \cdot \phi(d1) \cdot \frac{S}{K}$$

$$e^{-rT} \cdot K \cdot \phi(d_2) = S \cdot \phi(d_1)$$

Therefore, we have from equation (2) above that:

$$\frac{\partial C}{\partial S} = \Phi(d_1) + \frac{1}{S} \cdot \frac{1}{\sigma\sqrt{T}}[S \cdot \phi(d_1) - e^{-rT} \cdot K \cdot \phi(d_2)]$$

$$= \Phi(d_1) + \frac{1}{S} \cdot \frac{1}{\sigma\sqrt{T}}[S \cdot \phi(d_1) - S \cdot \phi(d_1)]$$

$$\frac{\partial C}{\partial S} = \Phi(d_1)$$

And we have thus verified the well known property of Black-Scholes; namely that $\Delta = \frac{\partial C}{\partial S} = \Phi(d_1)$.

This in turn yields a nice interpretation of the first term in the Black-Scholes formula in equation (1). That is $S \cdot \Phi(d_1)$ is the value of the long position in the stock required to replicate the European call option. Note that Δ is a function, not a constant.