Deriving the Black-Scholes PDE For a Dividend Paying Underlying Using a Hedging Portfolio

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3/19/2007

1 Set Up

The foundation of the Black-Scholes problem is modeling the stochastic stock process as Geometric Brownian Motion (GBM). In this case we have a stock that pays a dividend. Written in SDE form we have:

$$dS(t) = S(t)[(\mu + \delta)dt + \sigma dW(t)] \tag{1}$$

$$S(0) = s \tag{2}$$

Where μ is the mean return on the stock process, δ is the continuous dividend rate, σ is the volatility and W(t) is the standard Brownian Motion. One of the crucial assumptions to general Black-Scholes theory is the assumption that μ, δ and σ are constants. As we will see in the derivation, the "magic" of Black-Scholes allows us to price an option without using the mean return.

We define the dividend process D(t) as:

$$D(t) = \delta S(t) \tag{3}$$

$$dD(t) = \delta S(t)dt \tag{4}$$

The last piece of information we need to set up the problem is the movement of deterministic processes. Specifically, we define a hedging portfolio π(t) which we will construct to be entirely self financing and thus deterministic (non stochastic). In this framework, our deterministic processes satisfy the following differential equation:

$$d\pi(t) = r\pi(t)dt \tag{5}$$

Where r is the risk-free interest rate (assumed to be constant in this setting).

2 Creating the Hedging Portfolio and Deriving the BS PDE

In order to price the option, we need to construct a portfolio which will hedge the option exactly. We take the point of view of the seller of the option (short the option). We do this by creating a portfolio which is long Δ shares of stock (where Δ is to be determined) and short the option. With the correct choice of Δ we can make this portfolio deterministic (non stochastic) and self replicating. Note that Δ is not a constant but for notational convenience we omit the argument (t). We denote the option price as a function C(t,T, S(t), σ, r) and for short hand notation simply denoted as C.

$$\pi(t) = \Delta S(t) - C \tag{6}$$

The hedging portfolio changes in value by Δ times the stock process, plus Δ times the continuous dividend rate, minus the change in the call option. In English, this means that the portfolio the option seller holds moves up and down based on the stock price, the dividends received on the stock shares owned and the value of the option. This yields the SDE:

$$d\pi(t) = \Delta(dS(t) + dD(t)) - dC \tag{7}$$

We apply Ito's formula (with the subscript notation denoting partial derivatives) to the Call option and expand to get the following:

$$dC = C_t dt + C_s dS(t) + \frac{1}{2}C_{ss}(dS(t))^2 \tag{8}$$

$$= C_t dt + C_s S(t)[(\mu + \delta)dt + \sigma dW(t)] + \frac{1}{2}C_{ss}(S(t)[(\mu + \delta)dt + \sigma dW(t)]^2 \tag{9}$$

$$dC = C_t dt + C_s S(t)[(\mu + \delta)dt + \sigma dW(t)] + \frac{1}{2}C_{ss}(S(t))^2\sigma^2 dt \tag{10}$$

Now plugging dC into the equation for dπ(t) we get:

$$d\pi(t) = \Delta(dS(t) + dD(t)) - C_t dt - C_s S(t)[(\mu + \delta)dt - \sigma dW(t)] - \frac{1}{2}C_{ss}(S(t))^2\sigma^2 dt \tag{11}$$

$$= \Delta S(t)[(\mu+\delta)dt+\sigma dW(t)]+\Delta S(t)\delta dt - C_t dt - C_s S(t)[(\mu+\delta)dt-\sigma dW(t)] - \frac{1}{2}C_{ss}(S(t))^2\sigma^2 dt \tag{12}$$

We notice now that dπ(t) has stochastic terms. In order to remove the hedging portfolio of any stochastic components we can select the appropriate Δ, recalling that Δ is the number of shares we want to be long in the stock. In order for the *dW(t)* terms to disappear, we isolate the *dW(t)* terms and set them equal to each other. This yields:

$$\Delta S(t)\sigma = C_s S(t)\sigma \tag{13}$$

And finally solving for Δ we find:

$$\Delta = C_s \tag{14}$$

Now, replacing Δ into equation (12) and simplifying we get:

$$d\pi(t) = C_s S(t)\delta dt - C_t dt - \frac{1}{2}C_{ss}(S(t))^2\sigma^2 dt \tag{15}$$

And we note in the last step that by canceling the *dW(t)* terms we coincidentally cancel the μ terms which makes the Black-Scholes formulation so useful. We now set equation (15) with equation (5) and simply to get:

$$d\pi(t) = r\pi(t)dt = C_s S(t)\delta dt - C_t dt - \frac{1}{2}C_{ss}(S(t))^2\sigma^2 dt \tag{16}$$

$$r\pi(t) = C_s S(t)\delta - C_t - \frac{1}{2}C_{ss}(S(t))^2\sigma^2 \tag{17}$$

Plugging in equation (6) for $\pi(t)$ we get:

$$C_s[S(t) - C]r = C_s S(t)\delta - C_t - \frac{1}{2}C_{ss}(S(t))^2\sigma^2 \tag{18}$$

$$C_t + \frac{1}{2}C_{ss}(S(t))^2\sigma^2 + rC_s S(t) - C_s S(t)\delta - rC = 0 \tag{19}$$

$$C_t + \frac{1}{2}C_{ss}(S(t))^2\sigma^2 + C_s S(t)[r - \delta] - rC = 0 \tag{20}$$

which is the desired Black-Scholes PDE for a European Call Option with underlying paying a dividend:

$$C_t + \frac{1}{2}C_{ss}S^2\sigma^2 + C_s S[r - \delta] - rC = 0 \tag{21}$$

With terminal condition determined by the option payoff:

$$C(T) = max(S(T) - K, 0);\ \ (t < T) \tag{22}$$